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Potential symmetries of nonlinear diffusion–convection equations

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Abstract. In this paper potential symmetries are sought for the nonlinear diffusion–convection equations $u_t = [f(u)u_x]_x - [k(u)]_x$. The functional forms of $f(u)$ and $k(u)$ that admit such symmetries are completely classified.

We consider the nonlinear diffusion–convection equations of the type

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[f(u)\frac{\partial u}{\partial x}\right] - \frac{\mathrm{d}k(u)}{\mathrm{d}u}\frac{\partial u}{\partial x} \tag{1}$$

which have a number of applications in the study of porous media [1–4]. There is continuing interest in finding exact solutions to these equations [5, 6]. A complete classification of the Lie point symmetries of equation (1) is presented in [7, 8].

Bluman *et al* [9, 10] introduced a method for finding a new class of symmetries for a system of partial differential equations (PDEs) $\Delta(x, u)$, in the case that at least one of the PDEs can be written in conserved form. If we introduce potential variables v for the PDEs written in conserved form as further unknown functions, we obtain a system $Z(x, u, v)$. Any Lie group of transformations for $Z(x, u, v)$ induces a symmetry for $\Delta(x, u)$. When at least one of the generators which correspond to the variables x and u depends explicitly on the potential v, then the local symmetry of $Z(x, u, v)$ induces a non-local symmetry of $\Delta(x, u)$. These non-local symmetries are called *potential symmetries*.

In the spirit of [8], where a complete group classification of point symmetries admitted by (1) is presented, we search for potential symmetries of (1). We classify all the functions $f(u)$ and $k(u)$ that admit such symmetries. Introducing the potential v, equation (1) can be written as a system of two PDEs:

$$v_x = u \qquad v_t = f(u)u_x - k(u). \tag{2}$$

We determine the infinitesimal transformations of the form

$$\begin{aligned} x' &= x + \epsilon X(x, t, u, v) + \mathrm{o}(\epsilon^2) \\ t' &= t + \epsilon T(x, t, u, v) + \mathrm{o}(\epsilon^2) \\ u' &= u + \epsilon U(x, t, u, v) + \mathrm{o}(\epsilon^2) \\ v' &= v + \epsilon V(x, t, u, v) + \mathrm{o}(\epsilon^2) \end{aligned} \tag{3}$$

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which are admitted by equations (2). These transformations induce potential and point symmetries for (1) and point symmetries for the integrated form of (1)

$$v_t = f(v_x)v_{xx} - k(v_x) \tag{4}$$

where $u = v_x$.

Equations (2) admit Lie transformations of the form (3) if and only if

$$\Gamma^{(1)}\{v_x - u\} = 0 \qquad \Gamma^{(1)}\{v_t - f(u)u_x + k(u)\} = 0 \tag{5}$$

where $\Gamma^{(1)}$ is the first extended generator of

$$\Gamma = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v}$$

which is given by the relation

$$\Gamma^{(1)} = \Gamma + [D_xU - (D_xX)u_x - (D_xT)u_t]\frac{\partial}{\partial u_x} + [D_tU - (D_tX)u_x - (D_tT)u_t]\frac{\partial}{\partial u_t}$$
$$+[D_xV - (D_xX)v_x - (D_xT)v_t]\frac{\partial}{\partial v_x} + [D_tV - (D_tX)v_x - (D_tT)v_t]\frac{\partial}{\partial v_t}.$$

Here D_x and D_t are the total derivatives with respect to x and t, respectively. Eliminating v_x and v_t from equations (2), equations (5) take the form

$$E_1(x, t, u, v, u_x, u_t) = 0 \qquad E_2(x, t, u, v, u_x, u_t) = 0 \tag{6}$$

where E_1 and E_2 are determined polynomials in u_x and u_t. We impose the condition that equations (6) are identities in six variables $x,\ t,\ u,\ v,\ u_x,\ u_t$ which are regarded as independent. These two identities enable the infinitesimal transformations to be derived and ultimately impose restrictions on the functional forms of $f,\ k,\ X,\ T,\ U$ and V.

Now we can successively calculate that $E_{1u_xu_x} = -fT_u$ and $E_{2u_t} - E_{1u_x} = 2f(uT_v + T_x)$. Hence, $T = T(t)$. In fact, it can be shown that when the first equation in (2) is of the form $v_x = L(x, t, u, u_x)$ then the generator T is a function of t only [11]. Calculation of $E_{2u_xu_x}$ and E_{1u_x}, respectively, give $X_u = V_u = 0$. From the first identity in (6) we have

$$U = -X_vu^2 + (V_v - X_x)u + V_x. \tag{7}$$

Finally, the coefficient of u_x and the term independent of u_x in $E_2 = 0$ lead to

$$\left[X_vu^2 + (X_x - V_v)u - V_x\right]\frac{\mathrm{d}f}{\mathrm{d}u} + [2X_vu + 2X_x - T_t]\,f = 0 \tag{8}$$

$$\left[X_vu^2 + (X_x - V_v)u - V_x\right]\frac{\mathrm{d}k}{\mathrm{d}u} + [-X_vu + V_v - T_t]\,k$$
$$= \left[X_{vv}u^3 + (2X_{xv} - V_{vv})u^2 + (X_{xx} - 2V_{xv})u - V_{xx}\right]f + V_t - X_tu. \tag{9}$$

These last two equations enable us to deduce the functional forms of $f(u)$ and $k(u)$ and to derive the generators $X,\ T$ and V. Furthermore, equation (7) provides us with the generator U. From equation (8), we conclude that the function $f(u)$ satisfies an ordinary differential equation (ODE) of the form

$$\left(\lambda_1u^2 + \lambda_2u + \lambda_3\right)\frac{\mathrm{d}f}{\mathrm{d}u} + (2\lambda_1u + \lambda_4)\,f = 0$$

where the λ_i are constants. Similarly, as in the case where k = constant (the nonlinear diffusion equation) [9,10], it can be shown that equation (1) admits a potential symmetry, corresponding to the auxiliary system (2), if and only if the function $f(u)$ is of the form

$$f(u) = \frac{1}{u^2 + pu + q}\exp\left[r\int\frac{\mathrm{d}u}{u^2 + pu + q}\right] \tag{10}$$

where $p = \lambda_2/\lambda_1$, $q = \lambda_3/\lambda_1$, and $r = (\lambda_4 - \lambda_2)/\lambda_1$. We also state that (1) admits potential symmetries when f = constant. Any other form of $f(u)$ which satisfies the above ODE will induce point symmetries of (1). These symmetries are presented in the appendix. In addition, from equation (9) we deduce that the function $k(u)$ satisfies the ODE

$$\left(\lambda_1 u^2 + \lambda_2 u + \lambda_3\right) \frac{dk}{du} + \left[-\lambda_1 u - \lambda_2 + \frac{1}{2}(\lambda_4 - \lambda_5)\right] k = \lambda_6 u + \lambda_7.$$

Solving the above ODE we obtain

$$\begin{aligned} &\frac{d}{du}[I(u)k(u)] = \left[\frac{\lambda_6 u + \lambda_7}{u^2 + pu + q}\right] I(u) \\ &I(u) = \left[\frac{1}{\sqrt{u^2 + pu + q}} \exp\left(s \int \frac{du}{u^2 + pu + q}\right)\right] \end{aligned} \tag{11}$$

where $s = (\lambda_4 - \lambda_2 - \lambda_5)/(2\lambda_1)$.

We now employ equations (7)–(11) to derive the desired potential symmetries. We split the analysis into three cases: (i) $f = p/(u+q)^2$, (ii) f is given by (10) with $p^2 - 4q - r^2 \neq 0$ and (iii) f = constant. The form of f in case (i) is obtained by setting $p^2 - 4q - r^2 = 0$ and the constants p, q are redefined in (10).

Case (i). $f = p/(u+q)^2$

The functional forms of $k(u)$ may be found from (11). We only present the forms which produce potential symmetries. We omit any further calculations, which have been greatly facilitated by the computer algebraic package *REDUCE* [12].

(a) $k = r(u+q)^m/(u+s)^{m-1}$, $(q \neq s)$. From equations (7)–(9) we obtain

$$\begin{aligned} &T = 2m(q-s)c_1 t + c_2 \qquad &&X = c_1((mq - ms - s)x - v) + c_3 \\ &U = c_1(u+q)(u+s) \qquad &&V = c_1(qsx + (mq - ms + q)v) + c_4. \end{aligned}$$

That is, equation (1) admits the potential symmetry

$$\begin{aligned} \Gamma_1 = {} & 2m(q-s)t\frac{\partial}{\partial t} + ((mq - ms - s)x - v)\frac{\partial}{\partial x} + (u+q)(u+s)\frac{\partial}{\partial u} \\ & + (qsx + (mq - ms + q)v)\frac{\partial}{\partial v}. \end{aligned}$$

(b) $k = r(u+q)\exp\left(s/(u+q)\right)$, $(s \neq 0)$. Here equation (1) admits the potential symmetry

$$\Gamma_2 = 2st\frac{\partial}{\partial t} + ((s-q)x - v)\frac{\partial}{\partial x} + (u+q)^2\frac{\partial}{\partial u} + (q^2 x + (q+s)v)\frac{\partial}{\partial v}.$$

(c) $k = r/(u+q)$. Equation (1) admits the following potential symmetries:

$$\begin{aligned} \Gamma_3 = {} & 4rt^2\frac{\partial}{\partial t} - [2pt + (v+qx)^2]\frac{\partial}{\partial x} + 2(u+q)\left[(u+q)(v+qx) + 2rt\right]\frac{\partial}{\partial u} \\ & + \left[q(v+qx)^2 + 4rt(v+qx) + 2pqt\right]\frac{\partial}{\partial v} \end{aligned}$$

$$\Gamma_4 = (v+qx)\frac{\partial}{\partial x} - (u+q)^2\frac{\partial}{\partial u} - [q(v+qx) + 2rt]\frac{\partial}{\partial v}$$

$$\Gamma_{1\infty} = \mathrm{e}^{-rx/p}\left[p\phi\frac{\partial}{\partial x} - (u+q)(p(u+q)\phi_\xi - r\phi)\frac{\partial}{\partial u} - pq\phi\frac{\partial}{\partial v}\right]$$

where in $\Gamma_{1\infty}$, $y = \phi(t,\xi)$, $\xi = v + qx$ is an arbitrary solution of the linear heat equation

$$p\frac{\partial^2 y}{\partial \xi^2} - \frac{\partial y}{\partial t} = 0. \tag{12}$$

In addition, equations (2) admit point symmetries which are projected onto point symmetries of (1) and (4). These symmetries are presented in the appendix.

(d) $k = r(u+q)$. We have the following potential symmetries:

$$\Gamma_5 = (v + qrt)\frac{\partial}{\partial x} - u(u+q)\frac{\partial}{\partial u} - q(v+qrt)\frac{\partial}{\partial v}$$

$$\begin{aligned}\Gamma_6 = {} & 12pqt^2\frac{\partial}{\partial t} + \left[(v+qx)^3 + 3q(rt-x)(v+qx)^2 + 6(ptv + 3pqrt^2)\right]\frac{\partial}{\partial x}\\ & +3(u+q)\left[-u(v+qx)^2 + 2q(u+q)(qx^2 - rtv - qrtx + xv)\right.\\ & \left. + 4pqt - 2ptu\right]\frac{\partial}{\partial u} + q\left[-(v+qx)^3 + 3q(x-rt)(v+qx)^2\right.\\ & \left. + 6(ptv + 2pqxt - 3pqrt^2)\right]\frac{\partial}{\partial v}\end{aligned}$$

$$\begin{aligned}\Gamma_7 = {} & \left[(v+qx)(v-qx) + 2(qrtv + pt + q^2rtx)\right]\frac{\partial}{\partial x}\\ & +2(u+q)\left[-uv + q^2x - qrt(u+q)\right]\frac{\partial}{\partial u}\\ & +q\left[-(v+qx)(v-qx) + 2(pt - qrtv - q^2rtx)\right]\frac{\partial}{\partial v}\end{aligned}$$

$$\Gamma_{2\infty} = \phi\frac{\partial}{\partial x} - (u+q)^2\phi_\xi\frac{\partial}{\partial u} - q\phi\frac{\partial}{\partial v}$$

where $y = \phi(t,\xi)$ satisfies (12).

Case (ii). $f = \left(1/(u^2+pu+q)\right)\exp\left[r\int \mathrm{d}u/(u^2+pu+q)\right]$, $(p^2 - 4q - r^2 \neq 0)$

Upon substitution the above form of $f(u)$ in equations (8) and (9), we deduce that X and V are linear in x and v. In this case we obtain the following results:

(a) $k = \sqrt{u^2+pu+q}\ \exp\left[s\int \mathrm{d}u/(u^2+pu+q)\right]$. Equation (1) admits the potential symmetry

$$\begin{aligned}\Gamma_8 = {} & (r+2s)t\frac{\partial}{\partial t} + [(r+s-p/2)x - v]\frac{\partial}{\partial x} + (u^2+pu+q)\frac{\partial}{\partial u}\\ & +[qx + (r+s+p/2)v]\frac{\partial}{\partial v}.\end{aligned}$$

(b) $k = \left(1/I(u)\right)\int\left[(\lambda_1 u + \lambda_2)/(u^2 + pu + q)\right]I(u)\,\mathrm{d}u$. Here the function $I(u)$ is given by (11). For this case we have

$$\Gamma_9 = (r+2s)t\frac{\partial}{\partial t} + [(r+s-p/2)x + \lambda_1 t - v]\frac{\partial}{\partial x} + (u^2+pu+q)\frac{\partial}{\partial u} + [qx - \lambda_2 t + (r+s+p/2)v]\frac{\partial}{\partial v}.$$

We note that if $\lambda_1 = \lambda_2 = 0$ then $\Gamma_9 \equiv \Gamma_8$.

(c) $k = \lambda(u+q)$. Equation (1) admits the following potential symmetries:

$$\Gamma_{10} = (p^2 - 4q - r^2)t\frac{\partial}{\partial t} - \left[(p+r)v + 2qx + \lambda(r^2 + pq + qr + 2q - p^2)t\right]\frac{\partial}{\partial x} + (p+r)(u^2+pu+q)\frac{\partial}{\partial u} + \left[(p^2 + pr - 2q)v + q(p+r)x + \lambda q(pr - p + 2q + r^2 - r)t\right]\frac{\partial}{\partial v}$$

$$\Gamma_{11} = [2v + (p-r)x + \lambda(2q - p + r)t]\frac{\partial}{\partial x} - 2(u^2+pu+q)\frac{\partial}{\partial u} - \left[(p+r)v + 2qx + \lambda q(p+r-2)t\right]\frac{\partial}{\partial v}.$$

Case (iii). f = constant = p

From equation (8) we get $X = \frac{1}{2}xT_t + \theta(t)$ and from (9) we deduce that the function $k(u)$ satisfies an ODE of the form

$$(\lambda_1 u + \lambda_2)\frac{\mathrm{d}k}{\mathrm{d}u} + \lambda_3 k = \lambda_4 u^2 + \lambda_5 u + \lambda_6.$$

Equation (1) admits a potential symmetry only when $k = r(u+s)^2$. Any other form of $k(u)$ which satisfies the above ODE leads to point symmetries (see appendix). We note that if $k = r(u+s)^2$ then equation (1) becomes the well known Burgers' equation which admits [10] the potential symmetry

$$\Gamma_{3\infty} = \mathrm{e}^{rv/p}(ph_x + rhu)\frac{\partial}{\partial u} + p\mathrm{e}^{rv/p}h\frac{\partial}{\partial v}$$

where the function $h(x,t)$ satisfies the linear PDE

$$h_t - ph_{xx} + 2rsh_x - \frac{r^2 s^2}{p}h = 0.$$

As in the case of point symmetries, potential symmetries may be used to derive similarity transformations (solutions). Such transformations reduce the number of independent variables of a system of partial differential equations by one. We shall present the similarity solutions which are obtained using the potential symmetries Γ_1 and Γ_2.

We consider the point symmetry Γ_1 of (2) which is a potential symmetry of (1), with $f(u) = p/(u+q)^2$ and $k(u) = r(u+q)^m/(u+s)^{m-1}$. The corresponding invariant surface conditions are

$$2m(q-s)tu_t + ((mq - ms - s)x - v)u_x = (u+q)(u+s)$$

$$2m(q-s)tv_t + ((mq - ms - s)x - v)v_x = qsx + (mq - ms + q)v$$

which admit the following three integrals:

$$c_1 = (v + qx)t^{-\frac{1}{2}} \qquad c_2 = \frac{v + sx}{q - s} t^{-(m+1)/2m} \qquad c_3 = \left(\frac{u + s}{u + q}\right) t^{-1/2m}.$$

From the above relations we derive the similarity solutions

$$u = \frac{qt^{1/2m} F_1(\eta) - s}{1 - t^{1/2m} F_1(\eta)} \qquad v = -s\eta t^{\frac{1}{2}} + qt^{(m+1)/2m} F_2(\eta) \tag{13}$$

where η is the similarity variable defined implicitly by the relation

$$\eta = xt^{-\frac{1}{2}} + t^{1/2m} F_2(\eta). \tag{14}$$

Upon substitution of (13) into the system (2) we obtain the system of ordinary differential equations

$$\frac{\mathrm{d}F_2}{\mathrm{d}\eta} = F_1 \qquad -\eta \frac{\mathrm{d}F_2}{\mathrm{d}\eta} + \left(1 + \frac{1}{m}\right) F_2 = \frac{2p}{(q - s)^2} \frac{\mathrm{d}F_1}{\mathrm{d}\eta} - 2rF_1^{1-m} \tag{15}$$

where the independent variable η is defined by relation (14). Employment of the solution of the system (15), (14) and the first relation in equations (13) will produce a similarity solution of (1).

In [11] it is pointed out that a wider class of similarity solution may be obtained by direct introduction of equations (13) in (1). We can therefore substitute the first relation in equations (13) in (1). In this way, we obtain a relation involving $\eta,\ F_1,\ F_2$, the derivatives of $F_1,\ F_2$ and t which appears as a parameter. Imposing the condition that this relation is identically zero for any value of the parameter t, will result in the system of ordinary differential equations

$$\mu F_1'' + \frac{1}{2}\eta F_1' + (m - 1)rF_1^{-m} - \frac{1}{2m} F_1 = 0$$

$$\mu(2F_1 + F_2')F_1'' - \mu F_1' F_2'' + 2r(m - 1)F_2' F_1^{-m} + \frac{1}{2m}(m + 1)F_2 F_1' \frac{3}{2m} F_1 F_2'$$
$$+\eta F_1' F_2' + mrF_1^{1-m} = 0$$

$$\mu(2F_1 + F_2')F_1 F_1'' - \mu F_1 F_1' F_2'' + r(m - 1)F_2'^2 F_1^{-m} + \frac{1}{m}(m + 1)F_2 F_2' F_1' \frac{3}{2m} F_1 F_2'^2$$
$$+\frac{1}{2}\eta F_1' F_2'^2 + 2rmF_2' F_1^{1-m} = 0$$

$$\mu F_1^2 F_2' F_1'' - 2\mu F_1^2 F_1' F_2'' - \frac{1}{2m} F_1 F_2'^3 + \frac{1}{2m}(m + 1)F_1' F_2 F_2'^2 + mrF_2'^2 F_1^{1-m} = 0$$

where $\mu = p/(q - s)^2$ and the primes indicate derivatives with respect to η. The solution of the above system, as is pointed out in [11], will also contain the solution of the system (15).

Now we consider the symmetry Γ_2. The corresponding invariant surface conditions are

$$2stu_t + ((s - q)x - v)u_x = (u + q)^2$$

$$2stv_t + ((s - q)x - v)v_x = q^2 x + (q + s)v$$

which admit the following integrals:

$$c_1 = (v + qx)t^{-\frac{1}{2}} \qquad c_2 = t^{-\frac{1}{2}} v - \frac{q}{2s}(v + qx)t^{-\frac{1}{2}} \ln t \qquad c_3 = \frac{2s}{u + q} + \ln t.$$

From these integrals we obtain the similarity solutions

$$u + q = \frac{2s}{F_1(\eta) - \ln t} \qquad v = \frac{q}{2s}\eta t^{\frac{1}{2}} \ln t + t^{\frac{1}{2}} F_2(\eta). \tag{16}$$

where the similarity variable is defined implicitly by the relation

$$\eta \left(1 - \frac{q}{2s} \ln t\right) = qxt^{-\frac{1}{2}} + F_2(\eta). \tag{17}$$

Upon substitution of (16) in the system (2) we obtain the system of ordinary differential equations

$$2sF_2' + qF_1 = 2s \qquad -\eta F_2' + F_2 + \frac{q}{s}\eta = \frac{pq}{s}F_1' - 2rq\mathrm{e}^{F_1/2}. \tag{18}$$

Similarly, one can derive the similarity solutions which are produced by the potential symmetries Γ_3–Γ_{11}.

In [10] it is shown that an invertible mapping which transforms a nonlinear PDE into a linear PDE does not exist if the nonlinear PDE does not admit an infinite-parameter Lie group of contact transformations. Also such mappings do not exist for a nonlinear system of PDEs if the system does not admit an infinite-parameter Lie group of transformations. If such infinite-parameter groups exist then the nonlinear PDE (or the system of nonlinear PDEs) can be transformed into a linear PDE (or into a system of linear PDEs), provided that these groups satisfy certain criteria [10].

As we have seen, the auxiliary system of (1), given by equations (2), admits an infinite-parameter Lie group of point transformations in the cases where $f = p(u+q)^{-2}$, $k = r(u+q)^{-1}$, $(\Gamma_{1\infty})$, $f = p(u+q)^{-2}$, $k = r(u+q)$, $(\Gamma_{2\infty})$ and $f =$ constant, $k = r(u+s)^2$, $(\Gamma_{3\infty})$. Only symmetries $\Gamma_{1\infty}$ and $\Gamma_{3\infty}$ lead to invertible mappings for the system (2). In turn, these mappings lead to non-invertible mappings of (1).

The procedure for determining such invertible mappings is well explained in [10]. Employing the infinitesimal generator $\Gamma_{3\infty}$ leads to an invertible mapping that linearizes equation (2) which in turn leads to the non-invertible Hopf–Cole transformation which connects the Burgers's equation with the linear heat equation (12).

The infinite symmetry $\Gamma_{1\infty}$ leads to the invertible mapping

$$x' = v + qx \qquad t' = t \qquad u' = \frac{1}{r}\mathrm{e}^{rx/p} \qquad v' = \frac{1}{p}\frac{\mathrm{e}^{rx/p}}{u+q} \tag{19}$$

which transforms any solution $(u'(x',t'), v'(x',t'))$ of the linear system of PDEs

$$u'_{x'} = v' \qquad u'_{t'} = pv'_{x'} \tag{20}$$

into a solution $(u(x,t), v(x,t))$ of the nonlinear system

$$v_x = u \qquad v_t = -\frac{p}{(u+q)^2}u_x + \frac{r}{u+q}. \tag{21}$$

In turn this mapping leads to a non-invertible transformation which connects (1) with the linear heat equation (12) [3, 4].

Appendix

In addition to the point symmetries of equations (2) which induce potential symmetries of (1), equations (2) admit point symmetries which project to point symmetries of (1). If $f(u)$ and $k(u)$ are arbitrary functions, then equations (2) admit the symmetries

$$X_1 = \frac{\partial}{\partial t} \qquad X_2 = \frac{\partial}{\partial x} \qquad X_3 = \frac{\partial}{\partial v}.$$

Additional symmetries exist depending on the functional forms of f and k. These symmetries appear in table A1.

We state that the symmetries Γ_1–Γ_{11}, $\Gamma_{1\infty}$–$\Gamma_{3\infty}$ and X_1–X_{15} constitute the complete group classification of point symmetries admitted by equation (4).

Table A1. Additional symmetries.

$f(u)$	$k(u)$	Symmetries
Constant	$r(u+s)^m + \lambda(u+s)$	$X_4 = 2(m-1)t\frac{\partial}{\partial t} + (m-1)(x+\lambda t)\frac{\partial}{\partial x} - (u+s)\frac{\partial}{\partial u}$ $+[(m-2)v + (1-m)s\lambda t - sx]\frac{\partial}{\partial v}$
Constant	$re^{su} + \lambda u$	$X_5 = 2st\frac{\partial}{\partial t} + s(x+\lambda t)\frac{\partial}{\partial x} - \frac{\partial}{\partial u} + (sv - x + \lambda t)\frac{\partial}{\partial v}$
Constant	$r\ln(u+s) + \lambda u$	$X_6 = 2t\frac{\partial}{\partial t} + (x+\lambda t)\frac{\partial}{\partial x} + (u+s)\frac{\partial}{\partial u}$ $+(2v + sx - rt - \lambda st)\frac{\partial}{\partial v}$
Constant	$r(u+s)\ln(u+s) + \lambda(u+s)$	$X_7 = rt\frac{\partial}{\partial x} + (u+s)\frac{\partial}{\partial u} + (v + sx - rst)\frac{\partial}{\partial v}$
$p =$ constant	$r(u+s)^2$	$X_4,\ X_8 = 2rt\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + (x - 2rst)\frac{\partial}{\partial v}$ $X_9 = rt^2\frac{\partial}{\partial t} + rtx\frac{\partial}{\partial x} + [x/2 - r(u+s)t]\frac{\partial}{\partial u}$ $+(x^2/4 + pt/2 - rstx)\frac{\partial}{\partial v}$
pe^{qu}	$r(u+s)^2$	$X_{10} = qt\frac{\partial}{\partial t} + (qx + 2rt)\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + (qv + x - 2rst)\frac{\partial}{\partial v}$
pe^{qu}	$re^{su} + \lambda u$	$X_{11} = (q-2s)t\frac{\partial}{\partial t} + [(q-s)x - \lambda st]\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$ $+[(q-s)v + x - \lambda t]\frac{\partial}{\partial v}$
$p(u+q)^n$	$r(u+q)^m + \lambda(u+q)$	$X_{12} = (2m-n-2)t\frac{\partial}{\partial t} + [\lambda t(m-1) + (m-n-1)x]\frac{\partial}{\partial x}$ $-(u+q)\frac{\partial}{\partial u} + [(m-n-2)v - qx + \lambda q(1-m)t]\frac{\partial}{\partial v}$
$p(u+q)^n$	$r(u+q)$	$X_{12},\ X_{13} = n(x - rt)\frac{\partial}{\partial x} + 2(u+q)\frac{\partial}{\partial u}$ $+[(n+2)v + 2qx + nrqt]\frac{\partial}{\partial v}$
$p(u+q)^n$	$r(u+q)\ln(u+q) + \lambda(u+q)$	$X_{14} = nt\frac{\partial}{\partial t} + (nx + rt)\frac{\partial}{\partial x} + (u+q)\frac{\partial}{\partial u}$ $+[(n+1)v + qx - qrt]\frac{\partial}{\partial v}$
$p(u+q)^n$	$r\ln(u+q) + \lambda u$	$X_{15} = (n+2)t\frac{\partial}{\partial t} + [(n+1)x + \lambda t]\frac{\partial}{\partial x} + (u+q)\frac{\partial}{\partial u}$ $+[(n+2)v + qx - (r + \lambda q)t]\frac{\partial}{\partial v}$

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